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(p, q)-strings and new spacetime superalgebras

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Abstract: We present a geometric formulation of (p, q)-strings in which the $\mathrm{SL}(2, \mathbb{Z})$-doublet of the two-form gauge potentials is constructed as second order in the supersymmetric currents. The currents are constructed using a supergroup manifold corresponding to the (p, q)-string superalgebra, which contains fermionic generators in addition to the supercharges and transforms under the $\mathrm{SL}(2, \mathbb{Z})$. The properties of the superalgebra and the generalizations to higher p-branes are discussed.

Keywords: Sigma Models, p-branes, D-branes, String Duality.

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1. Introduction

It is now widely appreciated that the IIB superstring theory enjoys $\mathrm{SL}(2, \mathbb{Z})$-symmetry non-perturbatively. (p, q)-strings [1] has been studied [2, 3, 4] as a formulation in which the $\mathrm{SL}(2, \mathbb{Z})$-symmetry is manifest. On the other hand, a manifestly supersymmetric formulation of the Green-Schwarz superstring, based on a superalgebra which was obtained as the global limit [5] of a superalgebra found by Siegel [6], was given by Siegel [7]. The superalgebra is a generalization of super Poincaré algebra, and contains a new fermionic generator. Constructing a suitable set of supercurrents on the corresponding supergroup manifold, he wrote down the Wess-Zumino term of the Green-Schwarz superstring in a second-order expression of the supercurrents. Using this formulation, p-branes $(p > 1)$, found in the brane scan [8], were formulated by Bergshoeff and Sezgin [9]. Introducing a set of superalgebras, they wrote the Wess-Zumino terms for the p-branes, which are $(p+1)$-th order in the supercurrents. These p-brane $(p \geq 1)$ minimal superalgebras were further investigated in [11]. Following the same line, non-minimal IIA strings and IIB F- and D-strings were formulated in [10], using the IIA superalgebra and the F- and D-string superalgebra, respectively.

In this paper, we consider (p, q)-strings using this formulation. It is known [10] that the pullback to the F(D)-string worldsheet of the NS⊗NS(R⊗R) gauge potential can be obtained from the F(D)-string superalgebra. In order to construct (p, q)-strings, we need the pullback to the same worldsheet of the NS⊗NS and the R⊗R gauge potentials. We obtain the pullback to the F-string worldsheet of the

R⊗R gauge potential (or, the pullback to the D-string worldsheet of the NS⊗NS gauge potential), in section 3, in the following two ways: First, we consider a map which transforms the D-string superalgebra to the F-string superalgebra. Using this map, we obtain the pullback to the F-string worldsheet of the R⊗R gauge potential explicitly. Second, we construct the pullback to the F-string worldsheet of the R⊗R gauge potential, starting from a superalgebra which is obtained by interchanging fermionic generators, Σ^α and $\tilde{\Sigma}^\alpha$, in the D-string superalgebra. This is a nontrivial feature of our formulation which was suggested in [10]: Interchanging fermionic generators results in interchanging the F- and the D-string worldsheets (or, exchanging the R⊗R two-form gauge potential for the NS⊗NS one), In order to confirm this feature, we show that the pullback to the F-string worldsheet of the R⊗R gauge potential can be obtained from a superalgebra which is obtained by interchanging fermionic generators in the F-string superalgebra. After a quick review of the explicit construction of the SL$(2,\mathbb{Z})$-doublet of two-form gauge potentials in terms of b_R and b_{NS}, we present the (p,q)-string superalgebra, and show that the SL$(2,\mathbb{Z})$-doublet of two-form gauge potentials can be constructed by using the superalgebra, in section 4. It is observed that at a special point on the SL$(2,\mathbb{Z})$-orbit, $\chi=\phi=0$, the (p,q)-string superalgebra reduces to the IIB superalgebra which is T-dual to the IIA superalgebra in presence of F-string, D0- and D2-branes. The last section is devoted to a summary and discussions of generalizations to the D3-brane, the (p,q) 5-brane and the KK5-brane.

2. F- and D-strings

We first review the formulation of IIB F- and D-strings following [10]. The F-string superalgebra is generated by supertranslations and F-string charges $T_A=\{Q_A, Z^A\}$, where $Q_A=(P_a, Q_\alpha, \tilde{Q}_\alpha)$ and $Z^A=(Z^a, Z^\alpha, \tilde{Z}^\alpha)$, as

$$\begin{aligned}
\{Q_{\alpha I}, Q_{\beta J}\} &= (\gamma^a \otimes I_{IJ})_{\alpha\beta} P_a - (\gamma_a \otimes \sigma_{3IJ})_{\alpha\beta} Z^a \,, \\
[P_a, Q_{\beta I}] &= -2(\gamma_a \otimes \sigma_{3IJ})_{\alpha\beta} Z^{\alpha J} \,, \\
[Z^a, Q_{\beta I}] &= 2(\gamma^a \otimes I_{IJ})_{\alpha\beta} Z^{\alpha J} \,,
\end{aligned} \tag{2.1}$$

where we employed a compact notation, $Q_{\alpha I}=\begin{pmatrix}\tilde{Q}_\alpha \\ Q_\alpha\end{pmatrix}$, etc. The indices I and J are frequently omitted below. The left-invariant supergroup vielbeins ${L_M}^A$ and the pullback one-form L^A are obtained by the left-invariant Maurer-Cartan one-form,

$$U^{-1}dU = dZ^M {L_M}^A T_A = L^A T_A \,, \tag{2.2}$$

where Z^M are coordinates on the supergroup manifold, and U is a supergroup element. Parametrizing the supergroup manifold by

$$U = e^{z_a Z^a} e^{\tilde{\zeta}_\alpha \tilde{Z}^\alpha} e^{\zeta_\alpha Z^\alpha} e^{x^a P_a} e^{\tilde{\theta}^\alpha \tilde{Q}_\alpha} e^{\theta^\alpha Q_\alpha} \,, \tag{2.3}$$

the pullback left-invariant vielbeins of the supergroup manifold, L^A, were obtained in terms of coordinates on the supergroup manifold. Similarly, the right-invariant supergroup vielbeins $R_M{}^A$ are obtained by the right-invariant Maurer-Cartan one-form,

$$dUU^{-1} = dZ^M R_M{}^A T_A \,. \tag{2.4}$$

The rigid supersymmetry transformations are interpreted as being the part of an infinitesimal transformation, $\delta Z^M = \epsilon^A R_A{}^M$, characterized by ϵ^α, where $R_A{}^M$ is the inverse of $R_M{}^A$. The pullback left-invariant vielbeins, L^A, are invariant under the supersymmetry transformation, by definition.

The pullback to the F-string worldsheet of the NS⊗NS two-form gauge potential was defined as a second-order expression in supercurrents,

$$B_{NS} = \frac{1}{2}\left(L^a \wedge L_a + \frac{1}{4}L^\alpha \wedge L_\alpha + \frac{1}{4}\tilde{L}^\alpha \wedge \tilde{L}_\alpha\right). \tag{2.5}$$

This is identical, up to total derivative terms, to the conventional Wess-Zumino term,

$$b_{NS} = \frac{1}{2}\left[dx^a \wedge \left((\bar\theta\gamma_a d\theta) - (\bar{\tilde\theta}\gamma_a d\tilde\theta)\right) + \frac{1}{2}(\bar{\tilde\theta}\gamma^a d\tilde\theta) \wedge (\bar\theta\gamma_a d\theta)\right]. \tag{2.6}$$

We use the capital letter B as the supersymmetric form of the Wess-Zumino term b to avoid confusion.

In turn, the D-string superalgebra is generated by supertranslations and D-string charges $T_A = \{Q_A, \Sigma^A\}$, where $\Sigma^A = (\Sigma^a, \Sigma^\alpha, \tilde\Sigma^\alpha)$, as

$$\begin{aligned}
\{Q_\alpha, Q_\beta\} &= (\gamma^a \otimes I)_{\alpha\beta} P_a + (\gamma_a \otimes \sigma_1)_{\alpha\beta}\Sigma^a \,, \\
[P_a, Q_\beta] &= 2(\gamma_a \otimes I)_{\alpha\beta}\Sigma^\alpha \,, \\
[\Sigma^a, Q_\beta] &= 2(\gamma^a \otimes \sigma_1)_{\alpha\beta}\Sigma^\alpha \,.
\end{aligned} \tag{2.7}$$

Parametrizing the supergroup manifold by

$$U = e^{y_a\Sigma^a} e^{\tilde\phi_\alpha\tilde\Sigma^\alpha} e^{\phi_\alpha\Sigma^\alpha} e^{x^a P_a} e^{\tilde\theta^\alpha \tilde Q_\alpha} e^{\theta^\alpha Q_\alpha} \,, \tag{2.8}$$

the pullback vielbeins of the left-invariant supergroup were obtained. As before, these are superinvariant. The pullback to the D-string worldsheet of the R⊗R two-form gauge potential was written down, in terms of supercurrents, as

$$B_R = -\frac{1}{4}\left(L^\alpha \wedge \hat{L}_\alpha - \tilde{L}^\alpha \wedge \hat{\tilde{L}}_\alpha\right), \tag{2.9}$$

where a caret on L represents a left-invariant pullback vielbein corresponding to D-string generators Σ^A. This was found to be the well-known Wess-Zumino term for D-strings, up to total derivative terms,

$$b_R = \frac{1}{2}\left[dx^a \wedge \left((\bar\theta\gamma_a d\theta) - (\bar{\tilde\theta}\gamma_a d\tilde\theta)\right) + \frac{1}{2}(\bar{\tilde\theta}\gamma^a d\tilde\theta) \wedge (\bar\theta\gamma_a d\theta)\right], \tag{2.10}$$

which is identical to the right hand side (r.h.s.) of (2.6).

3. The NS⊗NS and the R⊗R gauge potentials on the worldsheet

In order to describe (p,q)-strings, we need the pullback to the *same* worldsheet of the NS⊗NS and the R⊗R two-form gauge potentials, b_{NS} and b_R. In this section, we obtain the pullback to the F-string worldsheet of the R⊗R two-form gauge potential (or, the pullback to the D-string worldsheet of the NS⊗NS gauge potential) in the following two ways: First, we consider a map which transforms the D-string superalgebra to the F-string superalgebra. Second, we modify the D-string superalgebra, and show that the two-form potential can be constructed by the supercurrents on the corresponding supermanifold.

We denote the pullback to the D-string worldsheet of the R⊗R gauge potential as 'b_R on the D-string worldsheet $\mathcal{W}_D$' and the pullback to the F-string worldsheet of the R⊗R gauge potential as 'b_R on the F-string worldsheet $\mathcal{W}_F$', for simplicity.

3.1 Mapping the D-string superalgebra to the F-string superalgebra

Let us consider a map which transforms the D-string superalgebra to the F-string superalgebra. The dual map transforms the b_R on the D-string supergroup manifold $\mathcal{M}_D$ to a two-form on the F-string supergroup manifold $\mathcal{M}_F$. We regard the pullback of the obtained two-form as the two-form on the F-string worldsheet, since we have regarded the pullback of the b_R on the D-string supergroup manifold as the b_R on the D-string worldsheet in the previous section. In this sense, the obtained two-form can be regarded as the b_R on the F-string worldsheet $\mathcal{W}_F$. There is another point of view, which regards the two-form on the F-string supergroup manifold as b_{NS}. The pullback map thus transforms b_R to b_{NS} on the D-string worldsheet $\mathcal{W}_D$. Here, we adopt the former viewpoint below.

The map is summarized as

$$\begin{array}{ccccccc} b_R \text{ on } \mathcal{W}_D & \mathcal{W}_D & \hookrightarrow & \mathcal{M}_D & \leftrightarrow & T_A = \{Q_A, \Sigma^A\} \\ \downarrow * & \uparrow & & \uparrow & & \downarrow \\ b_R \text{ on } \mathcal{W}_F & \mathcal{W}_F & \hookrightarrow & \mathcal{M}_F & \leftrightarrow & T'_A = \{Q_A{}', Z^{A\prime}\}\,, \end{array} \tag{3.1}$$

where $\mathcal{M}_D \ni (x^a, \theta^\alpha, \tilde{\theta}^\alpha, y_a, \phi_\alpha, \tilde{\phi}_\alpha)$ and $\mathcal{M}_F \ni (x^a, \theta^\alpha, \tilde{\theta}^\alpha, z_a, \zeta_\alpha, \tilde{\zeta}_\alpha)$ are the D-string and F-string supergroup manifolds, respectively. The $*$ means the pullback. From the above diagram, one finds that a map: b_R on $\mathcal{W}_D \xrightarrow{*} b_R$ on $\mathcal{W}_F$, is induced by a map: $T_A \to T_A{}'$. Thus, we first consider the map which transforms the generators of D-string superalgebra to those of F-string superalgebra. We find that the D-string superalgebra is mapped to the F-string superalgebra by

$$\begin{aligned} P_a{}' &= P_a\,, & Q_\alpha{}' &= \frac{\epsilon}{\sqrt{2}}(Q+\tilde{Q})\,, & \tilde{Q}_\alpha{}' &= \frac{\epsilon}{\sqrt{2}}(-Q+\tilde{Q})\,, \\ Z^{a\prime} &= \Sigma^a\,, & Z^{\alpha\prime} &= \frac{\epsilon}{\sqrt{2}}(\Sigma^\alpha+\tilde{\Sigma}^\alpha)\,, & \tilde{Z}^{\alpha\prime} &= \frac{\epsilon}{\sqrt{2}}(\Sigma^\alpha-\tilde{\Sigma}^\alpha)\,, \end{aligned}$$

where $\epsilon^2 = 1$. This is a part of the $SO(2)$ R-symmetry. We next consider a map, which transforms spacetime coordinates on $\mathcal{M}_F$ to those on $\mathcal{M}_D$. The corresponding map of spacetime coordinates, associated with the above map of generators, is defined by

$$e^{z_a{}' Z^{a\prime}} e^{\tilde{\zeta}'_\alpha \tilde{Z}^{\alpha\prime}} e^{\zeta_\alpha{}' Z^{\alpha\prime}} e^{x^{a\prime} P_a{}'} e^{\tilde{\theta}^{\alpha\prime} \tilde{Q}'_\alpha} e^{\theta^{\alpha\prime} Q_\alpha{}'} = e^{y_a \Sigma^a} e^{\tilde{\phi}_\alpha \tilde{\Sigma}^\alpha} e^{\phi_\alpha \Sigma^\alpha} e^{x^a P_a} e^{\tilde{\theta}^\alpha \tilde{Q}_\alpha} e^{\theta^\alpha Q_\alpha} . \quad (3.2)$$

After some algebra, we obtain the following transformations:

$$x^a = x^{a\prime} , \qquad \theta = \frac{\epsilon}{\sqrt{2}}(\theta' - \tilde{\theta}') , \qquad \tilde{\theta} = \frac{\epsilon}{\sqrt{2}}(\theta' + \tilde{\theta}') , \qquad y_a = z_a{}' + \frac{1}{2}(\bar{\tilde{\theta}}{}' \gamma_a \theta') , \quad (3.3)$$

where expressions for ϕ_α and $\tilde{\phi}_\alpha$ are omitted. Performing the above transformations on b_R on $\mathcal{W}_D$, (2.10), we obtain b_R on $\mathcal{W}_F$ (or, b_{NS} on $\mathcal{W}_D$) as (dropping primes)

$$\begin{aligned} b_R = -\frac{1}{2}\Big\{ & dx^a \wedge \left[\left(\bar{\theta}\gamma_a d\tilde{\theta}\right) + \left(\bar{\tilde{\theta}}\gamma_a d\theta\right)\right] + \\ & + \frac{1}{4}\left[\left(\bar{\theta}\gamma^a d\theta\right) + \left(\bar{\tilde{\theta}}\gamma^a d\tilde{\theta}\right)\right] \wedge \left[\left(\bar{\tilde{\theta}}\gamma_a d\theta\right) + \left(\bar{\theta}\gamma_a d\tilde{\theta}\right)\right] \Big\}. \end{aligned} \quad (3.4)$$

We now write two-form b_R on $\mathcal{W}_F$ in a manifestly supersymmetric form. To do this, we note that the left-invariant Maurer-Cartan one-form of the F-string and D-string superalgebra satisfies

$$L^A T_A = L^{A\prime} T_A{}' , \quad (3.5)$$

because of (3.2). This implies the following relations:

$$\begin{aligned} & L^a = L^{a\prime} , && L^\alpha = \frac{\epsilon}{\sqrt{2}}(L^{\alpha\prime} - \tilde{L}^{\alpha\prime}) , && \tilde{L}^\alpha = \frac{\epsilon}{\sqrt{2}}(L^{\alpha\prime} + \tilde{L}^{\alpha\prime}) , \\ & \hat{L}_a = L_a{}' , && \hat{L}_\alpha = \frac{\epsilon}{\sqrt{2}}(L_\alpha{}' + \tilde{L}^{\alpha\prime}) , && \hat{\tilde{L}}_\alpha = \frac{\epsilon}{\sqrt{2}}(L_\alpha{}' - \tilde{L}_\alpha{}') , \end{aligned} \quad (3.6)$$

where a caret represents left-invariant vielbeins corresponding to D-string charges. Using the above relations in the supersymmetric form $\hat{B}$, (2.9), of b_R on $\mathcal{W}_D$, we obtain the supersymmetric form of b_R on $\mathcal{W}_F$ as

$$B_R = \frac{1}{4}\left(L^{\alpha\prime} \wedge \tilde{L}_\alpha{}' - \tilde{L}^{\alpha\prime} \wedge L_\alpha{}'\right) . \quad (3.7)$$

This is found to be identical to (3.4), up to total derivative terms, as expected. In this way, we have constructed b_R on $\mathcal{W}_F$ explicitly. Similarly, b_{NS} on $\mathcal{W}_D$ is constructed from b_{NS} on $\mathcal{W}_F$, in the same manner. The obtained expression for b_{NS} on $\mathcal{W}_D$ is identical to the r.h.s. of (3.4).

3.2 Interchanging fermionic generators

We have obtained b_R on $\mathcal{W}_F$ by a map which relates the D-string superalgebra to the F-string superalgebra. Here, we consider another course to construct b_R on $\mathcal{W}_F$. It was suggested [10] that a superalgebra, which is obtained by interchanging Σ^α and $\tilde{\Sigma}^\alpha$ in the D-string superalgebra, describes b_{NS} on $\mathcal{W}_D$ (or, equivalently b_R on $\mathcal{W}_F$). In fact, using a superalgebra which is obtained by interchanging fermionic generators, Σ^α and $\tilde{\Sigma}^\alpha$, in the D-string superalgebra, we can construct B_{NS} on $\mathcal{W}_D$ (or, B_R on $\mathcal{W}_F$), which is identical to the r.h.s. of (3.4) up to total derivative terms.[1] This implies that interchanging fermionic generators, Σ^α and $\tilde{\Sigma}^\alpha$, corresponds to exchanging the R⊗R gauge potential for the NS⊗NS one (or exchanging the D-string worldsheet for the F-string worldsheet). In this way, we find that the pullback of the R⊗R and the NS⊗NS gauge potentials to the same worldsheet can be described by the F-string superalgebra and a superalgebra which is obtained by interchanging Σ^α and $\tilde{\Sigma}^\alpha$ in the D-string superalgebra.

In order to confirm further the non-trivial feature: interchanging fermionic generators results in exchanging the NS⊗NS two-form for the R⊗R one (or, interchanging the F- and the D-string worldsheet), we show that by starting with a superalgebra, which is obtained by interchanging fermionic generators, Z^α and $\tilde{Z}^\alpha$, in the F-string superalgebra (2.1), b_R on $\mathcal{W}_F$ (or b_{NS} on $\mathcal{W}_D$) can be constructed. The corresponding left-invariant pullback vielbeins are obtained as

$$L^\alpha = d\theta^\alpha, \quad \tilde{L}^\alpha = d\tilde{\theta}^\alpha, \quad L^a = dx^a + \frac{1}{2}(\bar{\theta}\gamma^a d\theta) + \frac{1}{2}(\bar{\tilde{\theta}}\gamma^a d\tilde{\theta}), \tag{3.8}$$

$$L_a = dz_a + \frac{1}{2}(\bar{\theta}\gamma_a d\theta) - \frac{1}{2}(\bar{\tilde{\theta}}\gamma_a d\tilde{\theta}), \tag{3.9}$$

$$\tilde{L}_\alpha = d\zeta_\alpha + 2dz_a(\bar{\theta}\gamma^a)_\alpha + 2dx^a(\bar{\theta}\gamma_a)_\alpha + \frac{2}{3}(\bar{\theta}\gamma^a d\theta)(\bar{\theta}\gamma_a)_\alpha, \tag{3.10}$$

$$L_\alpha = d\tilde{\zeta}_\alpha + 2dz_a(\bar{\tilde{\theta}}\gamma^a)_\alpha - 2dx^a(\bar{\tilde{\theta}}\gamma_a)_\alpha - \frac{2}{3}(\bar{\tilde{\theta}}\gamma^a d\tilde{\theta})(\bar{\tilde{\theta}}\gamma_a)_\alpha. \tag{3.11}$$

We find that the second-order expression,

$$-\frac{1}{4}(L^\alpha \wedge L_\alpha - \tilde{L}^\alpha \wedge \tilde{L}_\alpha) \tag{3.12}$$

is a supersymmetric form of b_R on $\mathcal{W}_F$, (3.4), as expected. In this way, we have seen that the interchanging Z^α and $\tilde{Z}^\alpha$ results in exchanging the R⊗R gauge potential for the NS⊗NS one (or exchanging the D-string worldsheet for the F-string worldsheet).

[1] This turns out to be identical to the two-form found in [10]. In fact, using a Fierz identity

$$(\bar{\theta}\gamma^a d\theta) \wedge (\bar{\tilde{\theta}}\gamma_a d\theta) = \frac{1}{3}(\bar{\theta}\gamma^a d\theta) \wedge (\bar{\theta}\gamma_a d\tilde{\theta}) - \frac{1}{3}d\Big((\bar{\theta}\gamma^a d\theta)(\bar{\tilde{\theta}}\gamma_a\theta)\Big),$$

the last term in (3.4) is rewritten, up to total derivative terms, as, $\frac{1}{3}(\bar{\theta}\gamma^a d\theta) \wedge (\bar{\theta}\gamma_a d\tilde{\theta}) + \frac{1}{3}(\bar{\tilde{\theta}}\gamma^a d\tilde{\theta}) \wedge (\bar{\tilde{\theta}}\gamma_a d\theta)$.

4. The (p, q)-string superalgebra

In the previous section, we find that the superalgebra,

$$
\begin{aligned}
\{Q_\alpha, Q_\beta\} &= (\gamma^a \otimes I)_{\alpha\beta} P_a + (\gamma_a \otimes \sigma_1)_{\alpha\beta} \Sigma^a - (\gamma_a \otimes \sigma_3)_{\alpha\beta} Z^a, && (4.1)\\
[P_a, Q_\beta] &= 2(\gamma_a \otimes \sigma_1)_{\alpha\beta} \Sigma^\alpha - 2(\gamma_a \otimes \sigma_3)_{\alpha\beta} Z^\alpha, && (4.2)\\
[Z^a, Q_\beta] &= 2(\gamma^a \otimes I)_{\alpha\beta} Z^\alpha, && (4.3)\\
[\Sigma^a, Q_\beta] &= 2(\gamma^a \otimes I)_{\alpha\beta} \Sigma^\alpha, && (4.4)
\end{aligned}
$$

describes the pullback to the worldsheet of the R⊗R and the NS⊗NS two-form gauge potentials. Now that we have obtained the NS⊗NS and the R⊗R gauge potentials on the same worldsheet, we consider the (p, q)-string superalgebra which describes the SL$(2, \mathbb{Z})$-doublet of the two-form gauge potentials.

To do this, we first describe the SL$(2, \mathbb{Z})$-doublet of the two-form gauge potentials in terms of b_{NS} and b_R on the worldsheet, explicitly. In order to describe the SL$(2, \mathbb{Z})$-symmetry, we introduce background scalars which belong to the coset SL$(2, \mathbb{Z})$/SO(2) or, equivalently, SU$(1, 1)$/U(1). The SL$(2, \mathbb{Z})$ acts from the left and the SO(2) from the right. These scalars are represented by a complex SU$(1, 1)$ doublet $\mathcal{U}^r (r = 1, 2)$. Using these doublet, the SU$(1, 1)$-invariant complex scalar density is defined by

$$\boldsymbol{\Psi} = \mathcal{U}^1 \hat{\Psi} + \mathcal{U}^2 \Psi\,, \tag{4.5}$$

where $\hat{\Psi}$ and Ψ form an SL$(2, \mathbb{Z})$-doublet. Eliminating an unphysical field by using local U(1) gauge symmetry, i.e. in the unitary gauge, this is rewritten as[2]

$$\boldsymbol{\Psi} = -ie^{\phi/2} e^{-i\vartheta} (\hat{\Psi} - \bar{\tau}\Psi)\,, \tag{4.6}$$

where $e^{i\vartheta} = (\bar{\tau} - i)/|\tau + i|$ and $\tau = \chi + ie^{-\phi}$. The $\hat{\Psi}$ and Ψ are linear combinations of $\hat{\Phi}$ and Φ with coefficients written in terms of χ and ϕ, where $\hat{\Phi}$ and Φ are the Hodge duals of the superinvariant modified two-form field strengths $\hat{\mathcal{F}} = d\hat{A} - b_R$ and $\mathcal{F} = dA - b_{NS}$, respectively. These coefficients can be determined by requiring that $\boldsymbol{\Psi}$ should not include χ and ϕ explicitly, and that $\hat{\Psi}$ and Ψ should be $\hat{\Phi}$ and Φ, respectively, when χ and ϕ are set to vanish. As a result, one finds that $\hat{\Psi}$ and Ψ are expressed as

$$
\begin{aligned}
\hat{\Psi} &= e^{\phi/2} \Re(\tau e^{i\vartheta}) \hat{\Phi} - e^{\phi/2} \Im(\tau e^{i\vartheta}) \Phi\,, && (4.7)\\
\Psi &= e^{\phi/2} \Re(e^{i\vartheta}) \hat{\Phi} - e^{\phi/2} \Im(e^{i\vartheta}) \Phi\,, && (4.8)
\end{aligned}
$$

[2] As in [12], (4.5) is rewritten, in the unitary gauge, as $\boldsymbol{\Psi} = f(\boldsymbol{\Phi} - B\boldsymbol{\Phi}^*)$, where $\boldsymbol{\Phi}$ and $\boldsymbol{\Phi}^*$ are SU$(1, 1)$-doublet. The f is defined by $f = (1 - BB^*)^{-1/2}$, and the B transforms linear fractionally under the finite SU$(1, 1)$ group action. The B has values restricted to the interior of the unit disc. The B is expressed in terms of τ as $B = (\bar{\tau} + i)/(\bar{\tau} - i)$, and $\boldsymbol{\Phi}$ is in terms of $\hat{\Psi}$ and Ψ as $\boldsymbol{\Phi} = \hat{\Psi} + i\Psi$.

where $\Re(\rho)$ and $\Im(\rho)$ denote the real and imaginary parts of ρ, respectively. The Ψ is invariant under the left action of $\mathrm{SL}(2,\mathbb{Z})$:

$$\tau \to \frac{a\tau+b}{c\tau+d}\,, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})\,, \tag{4.9}$$

while keeping $\hat{\Phi}$ and Φ invariant. $\hat{\Psi}$ and Ψ form an $\mathrm{SL}(2,\mathbb{Z})$-doublet up to the right action of $\mathrm{SO}(2)$.

We now consider the (p,q)-string superalgebra which describes the $\mathrm{SL}(2,\mathbb{Z})$-doublet of the two-form gauge potentials described above. We first present the (p,q)-string superalgebra and then show that the $\mathrm{SL}(2,\mathbb{Z})$-doublet can be constructed from it. The (p,q)-string superalgebra is generated by $T_A = (Q_A, Z^A, \Sigma^A)$ as

$$\{Q_\alpha, Q_\beta\} = (\gamma^a \otimes I)_{\alpha\beta} P_a + (\gamma_a \otimes \sigma_1)_{\alpha\beta}(\alpha\Sigma^a - \gamma Z^a) + (\gamma_a \otimes \sigma_3)_{\alpha\beta}(\beta\Sigma^a - \delta Z^a)\,, \tag{4.10}$$

$$[P_a, Q_\beta] = 2(\gamma_a \otimes \sigma_1)_{\alpha\beta}(\alpha\Sigma^\alpha - \gamma Z^\alpha) + 2(\gamma_a \otimes \sigma_3)_{\alpha\beta}(\beta\Sigma^\alpha - \delta Z^\alpha)\,, \tag{4.11}$$

$$[Z^a, Q_\beta] = 2(\gamma^a \otimes I)_{\alpha\beta} Z^\alpha\,, \tag{4.12}$$

$$[\Sigma^a, Q_\beta] = 2(\gamma^a \otimes I)_{\alpha\beta} \Sigma^\alpha\,, \tag{4.13}$$

where α, β, γ and δ are expressed in terms of χ and ϕ as

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = e^{\phi/2} \begin{pmatrix} \Re(\tau e^{i\vartheta}) & -\Im(\tau e^{i\vartheta}) \\ \Re(e^{i\vartheta}) & -\Im(e^{i\vartheta}) \end{pmatrix}. \tag{4.14}$$

Note that our superalgebra contains fermionic charges, Z^α and Σ^α, as well as bosonic charges, Z^a and Σ^a, and these charges transform under the $\mathrm{SL}(2,\mathbb{Z})$. The left-invariant vielbeins on the corresponding group manifold are calculated as

$$L^\alpha = d\theta^\alpha\,, \qquad L^a = dx^a - \frac{1}{2}(\bar{\theta}\gamma^a \otimes I d\theta)\,, \tag{4.15}$$

$$L_a = dz_a - \frac{1}{2}\Big(\gamma(\bar{\theta}\gamma_a \otimes \sigma_1 d\theta) + \delta(\bar{\theta}\gamma_a \otimes \sigma_3 d\theta)\Big)\,, \tag{4.16}$$

$$\hat{L}_a = dy_a + \frac{1}{2}\Big(\alpha(\bar{\theta}\gamma_a \otimes \sigma_1 d\theta) + \beta(\bar{\theta}\gamma_a \otimes \sigma_3 d\theta)\Big)\,, \tag{4.17}$$

$$\begin{aligned} L_\alpha = {} & 2dz_a(\bar{\theta}\gamma^a \otimes I)_\alpha + d\zeta_\alpha - 2dx^a\Big(\gamma(\bar{\theta}\gamma_a \otimes \sigma_1)_\alpha + \delta(\bar{\theta}\gamma_a \otimes \sigma_3)_\alpha\Big) - \\ & -\frac{1}{3}(\bar{\theta}\gamma^a \otimes I d\theta)\Big(\gamma(\bar{\theta}\gamma_a \otimes \sigma_1)_\alpha + \delta(\bar{\theta}\gamma_a \otimes \sigma_3)_\alpha\Big) - \\ & -\frac{1}{3}\Big(\gamma(\bar{\theta}\gamma_a \otimes \sigma_1 d\theta) + \delta(\bar{\theta}\gamma_a \otimes \sigma_3 d\theta)\Big)(\bar{\theta}\gamma^a \otimes I)_\alpha\,, \end{aligned} \tag{4.18}$$

$$\begin{aligned} \hat{L}_\alpha = {} & 2dy_a(\bar{\theta}\gamma^a \otimes I)_\alpha + d\phi_\alpha + 2dx^a\Big(\alpha(\bar{\theta}\gamma_a \otimes \sigma_1)_\alpha + \beta(\bar{\theta}\gamma_a \otimes \sigma_3)_\alpha\Big) + \\ & +\frac{1}{3}(\bar{\theta}\gamma^a \otimes I d\theta)\Big(\alpha(\bar{\theta}\gamma_a \otimes \sigma_1)_\alpha + \beta(\bar{\theta}\gamma_a \otimes \sigma_3)_\alpha\Big) + \\ & +\frac{1}{3}\Big(\alpha(\bar{\theta}\gamma_a \otimes \sigma_1 d\theta) + \beta(\bar{\theta}\gamma_a \otimes \sigma_3 d\theta)\Big)(\bar{\theta}\gamma^a \otimes I)_\alpha\,. \end{aligned} \tag{4.19}$$

Using these supercurrents, we find that the supersymmetric form of the SL$(2,\mathbb{Z})$-doublet, $\hat{B}$ and B, are constructed as

$$\hat{B} = -\frac{1}{2}\left(L^a \wedge \hat{L}_a + \frac{1}{4}L^\alpha \wedge \hat{L}_\alpha\right), \tag{4.20}$$

$$B = \frac{1}{2}\left(L^a \wedge L_a + \frac{1}{4}L^\alpha \wedge L_\alpha\right). \tag{4.21}$$

In fact, these turn out to be

$$\hat{b} = \alpha b_R + \beta b_{NS}, \tag{4.22}$$

$$b = \gamma b_R + \delta b_{NS}, \tag{4.23}$$

up to total derivative terms, where b_{NS} and b_R are defined in (2.6) and (3.4), respectively. For constant dilaton and axion, the Hodge duals of the modified field strengths constructed from the above SL$(2,\mathbb{Z})$-doublet, $\hat{b}$ and b, are $\alpha\hat{\Phi} + \beta\Phi$ and $\gamma\hat{\Phi} + \delta\Phi$, which are nothing but (4.7) and (4.8), respectively. We thus regard the superalgebra (4.10)–(4.13) as the SL$(2,\mathbb{Z})$-*covariant* (p,q)-string superalgebra. At a special point on the SL$(2,\mathbb{Z})$-orbit, $\chi = \phi = 0$, this superalgebra reduces to (4.1)–(4.4), which is T-dual to the IIA superalgebra in presence of F-string, D0 and D2-branes [10].

R-symmetry of the superalgebra is realized as the right action of SO(2). In fact, the superalgebra is found to be invariant (with a trivial scaling of generators Σ^A and Z^A) under R-symmetry transformation: $Q_\alpha \to RQ_\alpha$, $\Sigma^\alpha \to R\Sigma^\alpha$ and $Z^\alpha \to RZ^\alpha$, where $R \in$ SO(2), associated with the right action of SO(2). In the case that $\chi = \phi = 0$, the SO(2) gauge transformation causes a rotation of Σ^A and Z^A. Thus, interchanging Σ^A and Z^A is a part of R-symmetry. This is similar to the case encountered in the map (3.2).

5. Summary and discussions

We obtained the pullback to the F-string worldsheet of the R⊗R gauge potential b_R by considering a map, which transforms the F-string superalgebra to the D-string superalgebra, or starting from a superalgebra, which is obtained by interchanging Σ^α and $\tilde{\Sigma}^\alpha$ in the D-string superalgebra. Considering the SL$(2,\mathbb{Z})$-doublet of the two-form gauge potentials explicitly in terms of the obtained b_R and b_{NS}, we presented the SL$(2,\mathbb{Z})$-*covariant* (p,q)-string superalgebra. It was shown that the SL$(2,\mathbb{Z})$-doublet can be constructed by the (p,q)-string superalgebra. The superalgebra at a special point on the SL$(2,\mathbb{Z})$-orbit, $\chi = \phi = 0$, reduces to the IIB superalgebra [10] which is related by T-duality to the IIA superalgebra describing F-strings, D0- and D2-branes.

Let us comment on the generalization of our formulation to the D3-brane, the (p,q) 5-brane and the KK5-brane. The D3-brane superalgebra will be obtained as a T-dual to the IIA superalgebra in presence of the D2-brane Σ^{AB} and the D4-brane $\Sigma^{A_1\cdots A_4}$. In addition, one finds that the algebraic consistency requires the F-string

Z^A and the D0-brane Σ. The D3-brane superalgebra is thus generated by generators: not only the supertranslation Q_A and the D3-brane Σ^{ABC} but also the F-string Z_A and the D-string Σ^A. This is consistent with the fact that the field strength H of the four-form gauge potential C, which naturally couples a D3-brane, is expressed as $H = dC + \Im(\mathcal{C} \wedge \bar{\mathcal{H}})$, where $\mathcal{C}$ represents the SU(1,1)-invariant two-form gauge potential and $\mathcal{H}$ the field strength. We anticipate that the supersymmetric form of the four-form gauge potential C can be constructed by using the D3-brane superalgebra. It is a general feature of our superalgebras that the p-brane superalgebra contains the p'-brane superalgebra ($p' < p \leq 5$) as a subalgebra. Since the superalgebra, which is T-dual to the IIA superalgebra in presence of the F-string, the D0- and the D2-brane, naturally describes the pullback to the same worldsheet of the NS$\otimes$NS and the R$\otimes$R gauge potentials, the D3-brane superalgebra may describe the D3-brane on which the (p, q)-string sticks. If this is the case, it is interesting to study the effects of interchanging fermionic generators of the D3-brane charges as well as the (p, q)-string charges, and to examine how the intersecting branes can be described in our formulation.

In turn, the (p, q) 5-brane superalgebra is generated by generators: the D5-brane and the NS 5-brane in addition to those of the D3-brane superalgebra. The 126 bosonic D5-brane charges $\Sigma^{m_1\cdots m_5}$, $m_i = 1, \ldots, 9$, are related by a T-duality to the 70 D4-brane charges $\Sigma^{p_1\cdots p_4}$ and the 56 D6-brane charges $\Sigma^{0p_1\cdots p_3}$ as $\Sigma^{\natural p_1\cdots p_4} = \Sigma^{p_1\cdots p_4}$ and $\Sigma^{p_1\cdots p_5} \sim \Sigma^{0p_1\cdots p_3}$, respectively, where "$\sim$" denotes the equivalence modulo the self-duality relation and p_i runs from 1 to 9 except for $\natural$. Thus, the D5-brane charges are naturally expressed in terms of the charges $\Sigma^{i_1\cdots i_4}$ in the IIA superalgebra, or equivalently $Z^{\natural i_1\cdots i_4}$ in the M-algebra, where $\natural$ denotes the 11-th direction. Similarly, the 126 IIB NS5(KK5)-brane charges $Z_B^{m_1\cdots m_5}$ $(Z_B^{0m_1\cdots m_4})$ are related by a T-duality to the 70 IIA NS5(KK5)-brane charges $Z^{\natural p_1\cdots p_4}$ $(Z^{0p_1\cdots p_4})$ and the 56 IIA KK5(NS5)-brane charges $Z^{\natural 0p_1\cdots p_3}$ $(Z^{p_1\cdots p_5})$ as $Z_B^{\natural p_1\cdots p_4} = Z^{\natural p_1\cdots p_4}$ $(Z_B^{0p_1\cdots p_4} = Z^{0p_1\cdots p_4})$ and $Z_B^{p_1\cdots p_5} \sim Z^{\natural 0p_1\cdots p_3}(Z_B^{0\natural p_1\cdots p_3} \sim Z^{p_1\cdots p_5})$, respectively. It follows that the IIB NS5(KK5)-brane charges are expressed in terms of the charges $Z^{\natural i_1\cdots i_4}$ $(Z^{i_1\cdots i_5})$ in the IIA superalgebra and the M-algebra. But it is not clear how the charges with spinorial indices are related to the IIA superalgebra. The D5- and NS5-brane charges with spinorial indices, except for those with five spinorial indices, are expected to be determined by the charges $Z^{\natural A_1\ldots A_4}$ and $Z^{\natural A_1\ldots A_4}$) in the M-algebra, respectively. But the charges with five spinorial indices will not be related to the charges of the M-algebra. If this is the case, we must add these charges with five fermionic indices in order to construct the corresponding gauge potentials in the manifestly supersymmetric forms. In the case of the KK5-brane charges, the situation is more intricate since the T-dual of the corresponding IIA superalgebra is not rewritten in a covariant form. Thus, the charges constituting the (p, q)5-brane and KK5-brane superalgebra are not determined in terms of those of the IIA superalgebra completely, unlike the D3-brane superalgebra. We hope to report on these issues in the near future [14].

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